

# Canonical and symplectic analysis for three dimensional gravity without dynamics

Alberto Escalante\*

*Instituto de Física, Benemérita Universidad Autónoma de Puebla,  
Apartado Postal J-48 72570, Puebla Pue., México,*

H. Osmart Ochoa-Gutiérrez

*Facultad de Ciencias Físico Matemáticas,  
Benemérita Universidad Autónoma de Puebla,  
Apartado postal 1152, 72001 Puebla, Pue., México.*

(Dated: October 7, 2016)

In this paper a detailed Hamiltonian analysis of three-dimensional gravity without dynamics proposed by V. Hussain is performed. We report the complete structure of the constraints and the Dirac brackets are explicitly computed. In addition, the Faddeev-Jackiw symplectic approach is developed; we report the complete set of Faddeev-Jackiw constraints and the generalized brackets, then we show that the Dirac and the generalized Faddeev-Jackiw brackets coincide to each other. Finally, the similarities and advantages between Faddeev-Jackiw and Dirac's formalism are briefly discussed.

PACS numbers: 98.80.-k, 98.80.Cq

## I. INTRODUCTION

It is well-known that the quantum study of gravity developed in Loop Quantum Gravity [LQG] is based on a background independent and non-perturbative canonical formulation [1–6]. In fact, the support of LQG is the canonical quantization scheme developed by Dirac-Bergman [7]. Dirac's canonical formalism is a powerful approach, it allows us identify the physical degrees of freedom, the gauge transformations, the complete structure of the constraints and the obtention of the extended action, this information is useful because a strict study of the symmetries will allow us to have the best guideline to make the quantization. Nonetheless, if a complete Dirac's canonical analysis is performed, in general it is complicated to classify the constraints in first and second class; the classification of the constraints is an important step to perform because first class constraints are generators of gauge transformations and allow us identify observables and second class constraints allow us to construct the Dirac brackets. However, in spite of Dirac's framework is a powerful tool for analyzing constrained systems, the quantum canonical formulation of gravity has been a difficult task to perform. For that reason, the study of toy models with a similar canonical structure to that

---

\*Electronic address: aescalan@ifuap.buap.mx

present in gravity becomes to be an interesting topic. In this respect, there are several examples of toy models with a close relation with gravity just as topological theories [8–16] and models as those reported by V. Hussain where there is not dynamics [17, 18], this means, in the canonical formulation of those models there is not an analog to the Hamiltonian constraint that is present in real gravity theory [19]. In this respect, the study of toy models is a interesting subject because those are good laboratories for testing classical and quantum ideas that could be applied to General Relativity [GR].

Because of the explained above, in this paper the Faddeev-Jackiw analysis for three dimensional gravity without dynamics reported in [17] is performed. In fact, the FJ framework is an alternative approach for studying constrained systems [19–21], the degrees of freedom are identified by means the so-called symplectic variables and these variables allow us to construct a symplectic matrix. In this manner, in FJ scheme all relevant information is contained in the symplectic matrix. Furthermore, since the system under study is singular there will be constraints and the FJ approach has the advantage that all the constraints of the theory are at the same footing, namely, it is not necessary perform the classification of the constraints in primary, secondary, first class or second class such as in Dirac’s method is done. Moreover, in FJ approach also it is possible to obtain the gauge transformations of the theory, and by fixing the gauge a symplectic tensor is constructed, then from that symplectic tensor it is possible to identify the generalized FJ brackets; generalized FJ and Dirac’s brackets coincide to each other. However, just as it has been commented in [22] in order to compare the results of Dirac’s approach with the FJ ones, it is necessary develop a complete Dirac’s analysis, this is, it is mandatory to follow all Dirac’s steps. In this respect, in this paper we develop a complete Dirac’s analysis of the theory under study, then we compare the Dirac results with the FJ ones; we will conclude that the FJ framework is more economical than Dirac’s scheme.

The paper is organized as follows, in Section II a pure Dirac’s analysis of three dimensional gravity without dynamics is performed. We identify the complete structure of the constraints and the second class constraints are eliminated by introducing the Dirac brackets. In Section III we develop a complete FJ symplectic analysis of the theory under study. The FJ constraints and the symmetries of the theory just as the gauge transformations are identified, then by fixing the gauge a symplectic tensor is constructed. From the symplectic tensor the generalized FJ brackets are identified and we compare these brackets with the Dirac ones; we will show that the FJ and Dirac’s brackets coincide to each other. Finally we present the conclusions and prospects.

## II. HAMILTONIAN ANALYSIS

In this section a detailed canonical analysis for Husain’s model is performed. As we have commented previously if we wish compare the canonical approach with the symplectic method, then it is necessary to perform the Hamiltonian analysis by following all Dirac’s steps. In this manner, the

action of interest is given by [17]

$$S[e, \omega] = \int_{\mathcal{M}} \epsilon^{\alpha\beta\gamma} \left[ \omega_\alpha \partial_\beta \omega_\gamma + \lambda e_\alpha^i \partial_\beta e_\gamma^i + \lambda \epsilon_{ij} e_\alpha^i e_\beta^j \omega_\gamma \right]. \quad (1)$$

where  $e_\alpha^i$  is the zweibein representing the gravitational field and  $\omega_\alpha$  is a gauge field,  $\lambda$  is the cosmological constant that can be positive or negative (in this work we shall assume that  $\lambda$  is positive),  $x^\mu$  are the coordinates that label the points of the 3-dimensional manifold  $\mathcal{M}$ . In our notation, Greek letters are indices for the spacetime and run from 0 to 2, while the middle alphabet letters ( $i, j, k = 1, 2$ ) are associated with the internal group and it can be raised and lowered with the metric  $\eta^{ij}$  with signature  $(+, +)$ . Thus, assuming that the manifold  $\mathcal{M}$  is topologically  $\Sigma \times R$ , the 2+1 decomposition allow us identify the following Lagrangian density

$$\mathcal{L} = \epsilon^{0ab} \omega_b \dot{\omega}_a + \lambda \epsilon^{0ab} e_b^i \dot{e}_{ai} + \omega_0 \{ 2\epsilon^{0ab} \partial_a \omega_b + \lambda \epsilon^{0ab} \epsilon_{ij} e_a^i e_b^j \} + e_0^i \{ 2\lambda \epsilon^{0ab} \partial_a e_{bi} + 2\lambda \epsilon^{0ab} \epsilon_{ij} e_a^j \omega_b \}, \quad (2)$$

where  $a, b, e = 1, 2$ . The matrix elements of the Hessian given by

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^i) \partial(\partial_\mu e_\beta^i)}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^i) \partial(\partial_\mu \omega_\beta)}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \omega_\alpha) \partial(\partial_\mu \omega_\beta)}, \quad (3)$$

are identically zero, thus, we expect 9 primary constraints. In order to identify the constraints, the canonical formalism calls for the definition of the momenta  $(\Pi_i^a, \Pi_i^0, P^0, P^a)$  canonically conjugate to  $(e_a^i, e_0^i, \omega_0, \omega_a)$  are given by

$$\Pi_i^a = \frac{\delta \mathcal{L}}{\delta \dot{e}_\alpha^i}, \quad P^a = \frac{\delta \mathcal{L}}{\delta \dot{\omega}_\alpha}. \quad (4)$$

From the null vectors of the Hessian and the definition of the momenta, we identify the following 9 primary constraints

$$\begin{aligned} \phi_i^a &: \Pi_i^a - \lambda \epsilon^{0ab} e_{bi} \approx 0, \\ \phi^a &: P^a - \epsilon^{0ab} \omega_b \approx 0, \\ \phi_i^0 &: \Pi_i^0 \approx 0, \\ \phi^0 &: P^0 \approx 0. \end{aligned} \quad (5)$$

In this manner, the canonical Hamiltonian takes the form

$$\mathcal{H}_c = \int \{ e_0^i (2\partial_a \Pi_i^a + 2\lambda \epsilon_{ij} e_a^j P^a) + \omega_0 (2\partial_a P^a + \epsilon_{ij} e_a^i \Pi^{aj}) \} dx^2, \quad (6)$$

and the primary Hamiltonian is defined as

$$\mathcal{H}_p = \mathcal{H}_c + \int [\lambda_a^i \phi_i^a + \lambda_a \phi^a + u_0^i \phi_i^0 + u_0 \phi^0] dx^2, \quad (7)$$

where  $\lambda_a^i, \lambda_a, u_0^i, u_0$  are Lagrange multipliers enforcing the primary constraints. For this theory, the fundamental Poisson brackets of the theory are given by

$$\begin{aligned} \{e_\alpha^i(x), \Pi_j^\beta(y)\} &= \delta_j^i \delta_\alpha^\beta \delta^2(x-y), \\ \{\omega_\alpha(x), P^\beta(y)\} &= \delta_\alpha^\beta \delta^2(x-y). \end{aligned} \quad (8)$$

In order to observe if there are more constraints, we calculate consistency of the constraints and we obtain the following 3 secondary constraints

$$\begin{aligned}\psi_i &: \partial_a \Pi_i^a + \lambda \epsilon_{ij} e_a^j P^a \approx 0, \\ \psi &: \partial_a P^a + \frac{1}{2} \epsilon_{ij} e_a^i \Pi^{aj} \approx 0.\end{aligned}\quad (9)$$

Consistency requires conservation in time of the secondary constraints, however, for this theory there are not third constraints. Now we need to classify all the constraints in first class and second class. For this aim, we calculate the following 12 x 12 matrix whose entries are the Poisson brackets between the primary and secondary constraints, the non-zero brackets are given by

$$\begin{aligned}\{\phi_i^a, \phi_i^g\} &= -2\epsilon^{ag} \eta_{ij} \delta^2(x-y), \\ \{\phi_i^a(x), \psi_j\} &= \lambda \epsilon_{ij} P^a + \lambda \epsilon^{ag} \eta_{ij} \partial_g \delta^2(x-y), \\ \{\phi^a(x), \phi^b(y)\} &= -2\epsilon^{ab} \delta^2(x-y), \\ \{\phi^a(x), \psi_j\} &= -\lambda \epsilon^{ag} \epsilon_{jk} e_g^k \delta^2(x-y), \\ \{\phi^a, \psi(y)\} &= \epsilon^{ag} \partial_g \delta^2(x-y),\end{aligned}\quad (10)$$

this matrix has  $rank = 6$  and 6 null vectors. Thus we expect 6 first class constraints and 6 second class constraints. Under a complicated work, we find the complete structure of the first class constraints given by

$$\begin{aligned}\phi_i^0 &: \Pi_i^0 \approx 0, \\ \phi^0 &: P^0 \approx 0, \\ \gamma_i &= \partial_a \Pi_i^a + \lambda \epsilon_{ij} e_a^j P^a + \frac{1}{2} \epsilon_{ad} \epsilon_i^l P^d \phi_l^a - \frac{1}{2} \partial_a \phi_i^a - \frac{\lambda}{2} \epsilon_{ij} e_b^j \phi^b \approx 0, \\ \gamma &= \partial_a P^a + \frac{1}{2} \epsilon_{ij} e_a^i \Pi^{aj} - \frac{1}{4} \epsilon_{ad} \epsilon^{li} \Pi_l^d \phi_i^a + \frac{\lambda}{4} \epsilon^{li} e_{la} \phi_i^a - \frac{1}{2} \partial_b \phi^b \approx 0,\end{aligned}\quad (11)$$

where we can observe that  $\gamma$  is the equivalent to the Gauss constraint and  $\gamma_i$  is a Gauss-like constraint, there is not the analog to the Hamiltonian constraint that is present in gravity, in this sense there is not dynamics. In fact,  $\gamma$  generates abelian transformations on the  $\omega_a$  field and rotations on the  $e_\alpha^i$  field, we will observe this point below. Furthermore, there are the following second class constraints given by

$$\begin{aligned}\chi_i^a &: \Pi_i^a - \lambda \epsilon^{0ab} e_{bi} \approx 0, \\ \chi^a &: P^a - \epsilon^{0ab} \omega_b \approx 0.\end{aligned}\quad (12)$$

In this manner, the algebra between the constraints is given by

$$\begin{aligned}\{\gamma_i(x), \gamma_j(y)\} &= \frac{\lambda}{2} \epsilon_{ij} \gamma \delta^2(x-y), \\ \{\gamma_i(x), \gamma(y)\} &= -\frac{1}{2} \epsilon_{il} \gamma^l \delta^2(x-y), \\ \{\gamma(x), \gamma(y)\} &= 0, \\ \{\gamma_i(x), \chi^a(y)\} &= \frac{1}{2} \epsilon_i^l \chi_l^a, \\ \{\gamma_i(x), \chi_j^a(y)\} &= \frac{\lambda}{2} \epsilon_{ij} \chi^a,\end{aligned}\quad (13)$$

where we can see that the algebra between the constraints is closed as expected. It is important to comment that the identification of the structure of the constraints (11) is a difficult task to perform and it has not been reported in the literature. On the other hand, with the information obtained until now, we can construct the Dirac brackets. For this aim we shall construct the matrix whose elements are only the Poisson brackets between the second class constraints, namely,  $C_{\alpha\beta}(u, v) = \{\chi^\alpha(x), \chi^\beta(y)\}$

$$C_{\alpha\beta}(u, v) = \begin{pmatrix} -2\lambda\epsilon^{ag}\eta_{il} & 0 \\ 0 & -2\epsilon^{ab} \end{pmatrix} \delta^2(u - v), \quad (14)$$

and its inverse

$$C_{\alpha\beta}^{-1}(u, v) = \begin{pmatrix} \frac{1}{2\lambda}\epsilon_{ag}\eta^{li} & 0 \\ 0 & \frac{1}{2}\epsilon_{ab} \end{pmatrix} \delta^2(u - v). \quad (15)$$

Furthermore, the Dirac brackets among two functionals, say  $A, B$ , are expressed by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P - \int du dv \{A(x), \chi_\alpha(u)\} C_{\alpha\beta}^{-1} \{\chi_\beta(v), B(y)\}, \quad (16)$$

where  $\{A(x), B(y)\}_P$  is the usual Poisson bracket between the functionals  $A, B$  and  $\chi_\alpha(u), \chi_\beta(v)$  is the set of second class constraints. Hence, by using (15) and (16) we obtain the following Dirac's brackets of the theory

$$\begin{aligned} \{e_a^i(x), e_b^j(y)\}_D &= \frac{\epsilon_{ab}}{2\lambda} \eta^{ij} \delta^2(x - y), \\ \{\omega_a(x), \omega_b(y)\}_D &= \frac{1}{2} \epsilon_{ab} \delta^2(x - y), \\ \{e_a^i(x), \Pi_j^b(y)\}_D &= \frac{1}{2} \delta_a^b \delta_j^i \delta^2(x - y), \\ \{\Pi_i^a(x), \Pi_j^b(y)\}_D &= \frac{1}{2} \epsilon^{ab} \eta_{ij} \delta^2(x - y), \\ \{P^a(x), P^b(y)\}_D &= \frac{1}{2} \epsilon^{ab} \delta^2(x - y), \\ \{\omega_a(x), P^b(y)\}_D &= \frac{1}{2} \delta_a^b \delta^2(x - y), \end{aligned} \quad (17)$$

it is important to note that the algebra of the first class constraints under the Dirac brackets coincide with (13). Moreover, we define the following gauge generator

$$G = \int dx^2 [\Lambda^i \gamma_i + \theta \gamma], \quad (18)$$

thus the following gauge transformations arise

$$\begin{aligned} \delta e_a^i &= -\frac{1}{2} \partial_a \Lambda^i - \frac{\Lambda^k}{2} \epsilon_k^i \omega_a + \frac{\theta}{2} \epsilon_k^i e_a^k, \\ \delta \omega_a &= -\frac{1}{2} \partial_a \theta - \frac{1}{2} \Lambda^i \epsilon_{il} e_a^l, \end{aligned} \quad (19)$$

where we can observe that these gauge transformations correspond to those found in FJ formalism (see the section below). In this manner, our results complete those reported in the literature [17].

### III. FADDEEV-JACKIW SYMPLECTIC FRAMEWORK

Now, the theory will be analyzed by using the FJ symplectic formalism. For this aim, we write the Lagrangian (2) in the following form

$$\mathcal{L}^{(0)} = \epsilon^{ab}\omega_b\dot{\omega}_a + \lambda\epsilon^{ab}e_b^i\dot{e}_{ai} - V^{(0)}, \quad (20)$$

where  $V^{(0)} = -\omega_0\{2\epsilon^{ab}\partial_a\omega_b + \lambda\epsilon^{ab}\epsilon_{ij}e_a^ie_b^j\} - e_0^i\{2\lambda\epsilon^{ab}\partial_a e_{bi} + 2\lambda\epsilon^{ab}\epsilon_{ij}e_a^je_b^j\}$  is identified as the symplectic potential. From the symplectic Lagrangian (20) we identify the following symplectic variables given by  $\xi^{(0)} = (e_a^i, e_0^i, \omega_a, \omega_0)$  and the 1-forms  $a^{(0)} = (\lambda\epsilon^{ab}e_{bi}, 0, \epsilon^{ab}\omega_b, 0)$ . In this manner, the symplectic matrix given by  $f_{ij}^{(0)} = \frac{\delta a_j}{\delta \xi^i} - \frac{\delta a_i}{\delta \xi^j}$  takes the form

$$f_{ij}^{(0)} = \begin{pmatrix} 2\lambda\epsilon^{ag}\eta_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\epsilon^{ab} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y), \quad (21)$$

where we can observe that  $f_{ij}^{(0)}$  is singular. The null vectors of that matrix are given by  $\mathcal{V}_1^{(0)} = (0, v^{e_0^i}, 0, 0)$  and  $\mathcal{V}_2^{(0)} = (0, 0, 0, v^{\omega_0})$ , where  $v^{e_0^i}$  and  $v^{\omega_0}$  are arbitrary functions. Hence, from the null vectors we obtain the following FJ constraints [21]

$$\Omega_i^{(0)} = \int dx^2 V_1^i \frac{\delta}{\delta \xi^i} \int dy^2 V(\xi) = \epsilon^{ab}\partial_a e_{bi} + \epsilon^{ab}\epsilon_{ij}e_a^j\omega_b = 0, \quad (22)$$

$$\beta^{(0)} = \int dx^2 V_2^i \frac{\delta}{\delta \xi^i} \int dy^2 V(\xi) = \epsilon^{ab}\partial_a \omega_b + \frac{\lambda}{2}\epsilon^{ab}\epsilon_{ij}e_a^ie_b^j = 0, \quad (23)$$

we can observe that these constraints correspond to the secondary constraints obtained in Dirac's approach (see the previous section). Furthermore, we need to know if there are more FJ constraints. Hence, we calculate the following system [21, 22]

$$\bar{f}_{kj}\dot{\xi}^{(0)j} = Z_k(\xi), \quad (24)$$

where

$$\bar{f}_{kj} = \begin{pmatrix} f_{ij}^{(0)} \\ \frac{\delta \Omega_i^{(0)}}{\delta \xi^{(0)j}} \\ \frac{\delta \beta^{(0)}}{\delta \xi^{(0)j}} \end{pmatrix} \quad \text{and} \quad Z_k = \begin{pmatrix} \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^{(0)j}} \\ 0 \\ 0 \end{pmatrix}, \quad (25)$$

thus, we construct the following symplectic matrix

$$\bar{f}_{ij} = \begin{pmatrix} 2\lambda\epsilon^{ag}\eta_{il} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\epsilon^{ag} & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon^{ag}\eta_{il}\partial_a + \epsilon^{gb}\epsilon_{il}\omega_b & 0 & \epsilon^{ag}\epsilon_{ij}e_a^j & 0 \\ \lambda\epsilon^{ag}\epsilon_{il}e_a^i & 0 & \epsilon^{ag}\partial_a & 0 \end{pmatrix} \delta^2(x-y), \quad (26)$$

we observe that this matrix is not square, however has null vectors. The null vectors are given by  $\bar{\mathcal{V}}_1 = (\delta_k^l \partial_a v^k - \epsilon_k^l \omega_a v^k, 0, \lambda \epsilon_{lj} e_a^j v^l, 0, -2\lambda v^k, 0)$  and  $\bar{\mathcal{V}}_2 = (\epsilon_j^l e_a^j v^\lambda, 0, -\partial_a v^\lambda, 0, 0, -2v^\lambda)$ . On the other hand,  $Z_k$  is given by

$$\bar{Z}_k = \begin{pmatrix} \frac{\delta V}{\delta \xi^i} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\lambda\omega_0\epsilon^{ab}\epsilon_{lj}e_b^j + 2\lambda\epsilon^{ba}\partial_b e_{0l} - 2\lambda\epsilon^{ab}\epsilon_{il}\omega_b e_0^i \\ -\Omega_i^{(0)} \\ \epsilon^{ba}2\partial_b\omega_0 - 2\lambda e_0^i\epsilon^{ba}\epsilon_{ij}e_b^j \\ \beta^{(0)} \\ 0 \\ 0 \end{pmatrix}. \quad (27)$$

The contraction of the null vectors with  $\bar{Z}_k$  vanishes because of the constraints. For instance, from the contraction with the first null vector we obtain

$$\begin{aligned} \bar{\mathcal{V}}_1^\mu \bar{Z}_\mu &= 2\lambda\omega_0\epsilon_{lj}\{\epsilon^{ab}\partial_a e_b^j + \epsilon^{jk}\epsilon^{ab}e_{ka}\omega_b\}v^l + 2\lambda e_0^i\epsilon_{il}\{\epsilon^{ab}\partial_a\omega_b + \frac{\lambda}{2}\epsilon^{kj}\epsilon^{ab}e_{ak}e_{bj}\} \\ &= \omega_0\epsilon_{lj}\Omega^j v^l + 2\lambda e_0^i\epsilon_{il}\beta v^i = 0, \end{aligned} \quad (28)$$

from the contraction with the second null vector we obtain

$$\bar{\mathcal{V}}_2^\mu \bar{Z}_\mu = e_{0i}\epsilon^{ij}\Omega_j v^\lambda = 0, \quad (29)$$

that contraction vanishes as well. In this manner, there are not more FJ constraints. Hence, we add the FJ constraints to the symplectic Lagrangian by using Lagrange multipliers namely  $\alpha$ ,  $\zeta$ , and the new symplectic Lagrangian is given by

$$\mathcal{L}^{(1)} = \epsilon^{ab}\omega_b\dot{\omega}_a + \lambda\epsilon^{ab}e_{bi}\dot{e}_a^i - \left(\epsilon^{ab}\partial_a\omega_b + \frac{\lambda}{2}\epsilon^{ab}\epsilon_{ij}e_a^i e_b^j\right)\dot{\alpha} - (\epsilon^{ab}\partial_a e_{bi} + \epsilon^{ab}\epsilon_{ij}e_a^j\omega_b)\dot{\zeta}^i - \bar{V}^{(1)}, \quad (30)$$

where  $\bar{V}^{(1)} = \bar{V}|_{\Omega_i, \beta}^{(0)} = 0$  vanishes because of the general covariance of the theory. We can observe that  $\dot{\alpha} = \omega_0$  and  $\dot{\zeta}^i = e_0^i$  has been taken into the account. From the symplectic Lagrangian (30) we identify the following symplectic variables  $\xi^{(1)} = (e_a^i, \zeta^i, \omega_a, \alpha)$  and the 1-forms  $a^{(1)} = \left(\lambda\epsilon^{ab}e_{bi}, -(\epsilon^{ab}\partial_a e_{bi} + \epsilon^{ab}\epsilon_{ij}e_a^j\omega_b), \epsilon^{ab}\omega_b, -(\epsilon^{ab}\partial_a\omega_b + \frac{\lambda}{2}\epsilon^{ab}\epsilon_{ij}e_a^i e_b^j)\right)$ , where the new symplectic matrix has the following form

$$f_{ij}^{(1)} = \begin{pmatrix} 2\lambda\epsilon^{ag}\eta_{ij} & -\epsilon^{ag}\eta_{il}\partial_a - \epsilon^{gb}\epsilon_{il}\omega_b & 0 & -\lambda\epsilon^{ag}\epsilon_{il}e_a^i \\ \epsilon^{ag}\eta_{il}\partial_a + \epsilon^{gb}\epsilon_{il}\omega_b & 0 & \epsilon^{ag}\epsilon_{ij}e_a^j & 0 \\ 0 & -\epsilon^{ag}\epsilon_{ij}e_a^j & 2\epsilon^{ag} & -\epsilon^{ag}\partial_a \\ \lambda\epsilon^{ag}\epsilon_{il}e_a^i & 0 & \epsilon^{ag}\partial_a & 0 \end{pmatrix} \delta^2(x-y), \quad (31)$$

we can observe that the matrix is singular. In fact, this means that the system has a gauge symmetry and it is well-known that the null vectors of the matrix (31) are generators of that symmetry [20].

In fact, the null vectors of the matrix (31) are given by

$$\begin{aligned}\Gamma_1 &= \left( \frac{\theta}{2} \epsilon_k^i e_a^k, 0, -\frac{1}{2} \partial_a \theta, -\theta \right), \\ \Gamma_2 &= \left( -\frac{1}{2} \partial_a \Lambda^i - \frac{\Lambda^k}{2} \epsilon_k^i \omega_a, -\lambda \Lambda^i, -\frac{1}{2} \Lambda^i \epsilon_{il} e_a^l, 0 \right),\end{aligned}\quad (32)$$

where  $\Lambda^i$  and  $\theta$  are gauge parameters. By using these null vectors, we find the following gauge transformations of the theory

$$\begin{aligned}\delta e_a^i &= -\frac{1}{2} \partial_a \Lambda^i - \frac{\Lambda^k}{2} \epsilon_k^i \omega_a + \frac{\theta}{2} \epsilon_k^i e_a^k, \\ \delta \omega_a &= -\frac{1}{2} \partial_a \theta - \frac{1}{2} \Lambda^i \epsilon_{il} e_a^l,\end{aligned}\quad (33)$$

where we can observe that these transformations coincide with those found in the Dirac scheme. Furthermore, we have commented above that Hussain's theory is diffeomorphism covariant and a (analog) Hamiltonian constraint is not present in the theory [17]. In fact, we can attend those points by redefining the gauge parameters as  $\Lambda^i = 2e_a^i \tau^a$  and  $\theta = -2\omega_a \tau^a$ , hence the gauge transformations take the form

$$\begin{aligned}\delta e_a^i &= \mathcal{L}_\tau e_a^i + \tau^b [\partial_a e_b^i - \partial_b e_a^i] + \tau^b \epsilon^i{}_k [e_a^k \omega_b - e_b^k \omega_a], \\ \delta \omega_a &= \mathcal{L}_\tau \omega_a + \tau^b [\partial_a \omega_b - \partial_b \omega_a + \tau^b \epsilon_{il} e_b^i e_a^l],\end{aligned}\quad (34)$$

which correspond (on shell) to diffeomorphisms and it is an internal symmetry of the theory. In this manner, we have reproduced by other way the results reported in [17].

On the other hand, we have showed that there are not more FJ constraints and the theory has a gauge symmetry, therefore, in order to obtain a symplectic tensor we fixing the temporal gauge

$$\begin{aligned}e_0^i &= 0, \\ \omega_0 &= 0,\end{aligned}\quad (35)$$

this implies that  $\alpha = cte, \zeta^i = cte$ . In this manner, by adding the temporal gauge as constraints, the new symplectic Lagrangian is given by

$$\mathcal{L}^{(2)} = \epsilon^{ab} \omega_b \dot{\omega}_a + \lambda \epsilon^{ab} e_{bj} \dot{e}_a^i - \left( \epsilon^{ab} \partial_a \omega_b + \frac{\lambda}{2} \epsilon^{ab} \epsilon_{ij} e_a^i e_b^j - \rho \right) \dot{\alpha} - (\epsilon^{ab} \partial_a e_{bi} + \epsilon^{ab} \epsilon_{ij} e_a^i \omega_b - \sigma_i) \dot{\zeta}^i, \quad (36)$$

where we choose the following symplectic variables  $\xi^{(2)} = (e_a^i, \zeta^i, \omega_a, \alpha, \rho, \sigma_i)$  and the 1-forms  $a^{(2)} = \left( \lambda \epsilon^{ab} e_{bi}, -(\epsilon^{ab} \partial_a e_{bi} + \epsilon^{ab} \epsilon_{ij} e_a^i \omega_b - \sigma_i), \epsilon^{ab} \omega_b, -(\epsilon^{ab} \partial_a \omega_b + \frac{\lambda}{2} \epsilon^{ab} \epsilon_{ij} e_a^i e_b^j - \rho), 0, 0 \right)$ . Now, the symplectic matrix takes the following form



$$\begin{aligned}
f_{ij}^{(2)} = & \begin{pmatrix} 2\lambda\epsilon^{ag}\eta_{il} & -\epsilon^{ag}\eta_{il}\partial_a - \epsilon^{gb}\epsilon_{il}\omega_b & 0 & -\lambda\epsilon^{ag}\epsilon_{il}e_a^i & 0 & 0 \\ \epsilon^{ag}\eta_{il}\partial_a + \epsilon^{gb}\epsilon_{il}\omega_b & 0 & \epsilon^{ag}\epsilon_{ij}e_a^j & 0 & 0 & -\delta_j^i \\ 0 & -\epsilon^{ag}\epsilon_{ij}e_a^j & 2\epsilon^{ag} & -\epsilon^{ag}\partial_a & 0 & 0 \\ \lambda\epsilon^{ag}\epsilon_{il}e_a^i & 0 & \epsilon^{ag}\partial_a & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \delta_j^i & 0 & 0 & 0 & 0 \end{pmatrix} \\
& \times \delta^2(x-y), \\
\end{aligned} \tag{37}$$

we observe that this matrix is a symplectic tensor, and its inverse is given by

$$\begin{aligned}
f_{ij}^{(2)-1} = & \begin{pmatrix} \frac{1}{2\lambda}\epsilon^{ag}\eta^{ij} & 0 & 0 & 0 & -\frac{\epsilon_j^i}{2}e_a^j & \frac{1}{2\lambda}(-\delta_j^i\partial_a + \epsilon_j^i\omega_a) \\ 0 & 0 & 0 & 0 & 0 & \delta_j^i \\ 0 & 0 & \frac{1}{2}\epsilon_{ab} & 0 & -\frac{1}{2}\partial_a & \frac{1}{2}\epsilon_{ij}e_a^j \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\epsilon_j^i}{2}e_a^j & 0 & \frac{1}{2}\partial_a & -1 & 0 & \epsilon_{ij}\frac{\epsilon^{ab}}{2}\partial_a e_b^j \\ \frac{1}{2\lambda}(\delta_j^i\partial_a - \epsilon_j^i\omega_a) & -\delta_j^i & -\frac{1}{2}\epsilon_{ij}e_a^j & 0 & -\frac{\epsilon_{ij}\epsilon^{ab}}{2}\partial_a e_b^j & 0 \end{pmatrix} \delta^2(x-y). \\
\end{aligned} \tag{38}$$

Therefore, from the symplectic tensor (38) we can identify the generalized FJ brackets by means of

$$\{\xi_i^{(2)}(x), \xi_j^{(2)}(y)\}_{FD} = [f_{ij}^{(2)}(x, y)]^{-1}, \tag{39}$$

thus, the following generalized brackets arise

$$\begin{aligned}
\{e_a^i, e_b^j\}_{FJ} &= \frac{1}{2\lambda}\epsilon_{ab}\eta^{ij}\delta^2(x-y), \\
\{\omega_a, \omega_b\}_{FJ} &= \frac{1}{2}\epsilon_{ab}\delta^2(x-y), \\
\end{aligned} \tag{40}$$

we can observe that the Dirac brackets and the FJ ones coincide to each other.

#### IV. SUMMARY AND CONCLUSIONS

In this paper a detailed canonical and symplectic analysis for Husain's gravity has been performed. The complete structure of the Dirac constraints and the algebra between them has been reported, we eliminated the second class constraints by introducing the Dirac brackets, then have used a temporal gauge in order to construct the new Dirac's brackets. Furthermore, with respect to the symplectic method, we obtained the complete set of FJ constraints, the gauge transformations were found and the diffeomorphisms were reported as a internal symmetry of the theory, then by fixing the temporal gauge a symplectic tensor has been constructed. From the symplectic tensor the generalized FJ brackets were identified and we showed that Dirac's and FJ brackets coincide to each other. It is important to comment that in Dirac's formulation the classification of the constraints

in first class and second class is a difficult task, in FJ approach, however, the identification of the constraints is less complicated and there are present less constraints than Dirac's method. In this sense, the FJ formulation is more elegant and economical.

## Acknowledgements

This work was supported by CONACyT under Grant No.CB-2014-01/240781. We would like to R. Cartas-Fuentevilla for discussion on the subject and reading of the manuscript.

- 
- [1] C. Rovelli: Quantum Gravity. Cambridge University Press, Cambridge (2004)
  - [2] T. Thiemann: Modern Canonical Quantum General Relativity. Cambridge University Press, Cambridge (2007).
  - [3] C. Rovelli: Living. Rev.Rel.1:1,1998.
  - [4] T. Thiemann, Lect.NotesPhys.721:185-263, (2007).
  - [5] A. Ashtekar and J. Lewandowski, Class.Quant.Grav.21:R53, (2004).
  - [6] J. F. Barbero, AIP Conf.Proc.1023:3-33, (2008).
  - [7] D. M. Gitman and I. V. Tyutin, Quantization of Fields with Constraints, Springer Series in Nuclear and Particle Physics (Springer, 1990); A. Hanson, T. Regge and C. Teitelboim, Constrained Hamiltonian Systems (Accademia Nazionale dei Lincei, Roma, 1978).
  - [8] E. Witten, Nuclear Phys. B 311 (1988) 46.
  - [9] V. Bonzom, E.R. Livine, Classical Quantum Gravity 25 (2008) 195024.
  - [10] O. Chandia, J. Zanelli Phys. Rev. D55 (1997) 7580-7585.
  - [11] A. Escalante, L. Carbajal, Ann. Physics 326 (2011) 323?339.
  - [12] G. T. Horowitz, Commun. Math. Phys. 125 (1989) 417.
  - [13] G. T. Horowitz and M. Srednicki, Commun. Math. Phys. 130 (1990) 83.
  - [14] A. Escalante, I. Rubalcava-Garcia, Int. J. Geom. Methods Mod. Phys. 09 (2012) 1250053.
  - [15] A. Escalante and P. Cavildo-Sánchez, submitted to Annals of Physics, arXiv:1607.02206.
  - [16] Y. Gang Miao, J. Ge Zhou, Y. Yang Liu, Phys. Lett. B 323 (1994) 169?173.
  - [17] V. Husain, arXiv:hep-th/9204029.
  - [18] V. Husain and K. Kuchar, Phys. Rev. D42 (1990) 4070.
  - [19] L. D. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).
  - [20] E. M. C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, F.I. Takakura, L.M.V. Xavier, Modern Phys. Lett. A 23 (2008) 829; E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, F.I. Takakura, Internat. J. Modern Phys. A 22 (2007) 3605; E.M.C. Abreu, C. Neves, W. Oliveira, Internat. J. Modern Phys. A 21 (2008) 5329; C. Neves, W. Oliveira, D.C. Rodrigues, C. Wotzasek, Phys. Rev. D 69 (2004) 045016; J. Phys. A 3 (2004) 9303; C. Neves, C. Wotzasek, Internat. J. Modern Phys. A 17 (2002) 4025; C. Neves, W. Oliveira, Phys. Lett. A 321 (2004) 267; J.A. Garcia, J.M. Pons, Internat. J. Modern Phys. A 12 (1997) 451; E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, R.C.N. Silva, C. Wotzasek, Phys. Lett. A 374 (2010) 3603-3607.
  - [21] L. Liao, Y.C. Huang, Ann. Phys. 322, 2469-2484, (2007).

- [22] A. Escalante, J. Manuel-Cabrera, Ann. Physics. 343, 27-39, (2014) ; A. Escalante, M. Zárate, Ann. Physics. 353, 163-178, (2015) ; A. Escalante, J. Manuel-Cabrera, Ann. Physics. 36, 1585-604, (2015).